Symmetry breaking and the f^2RG

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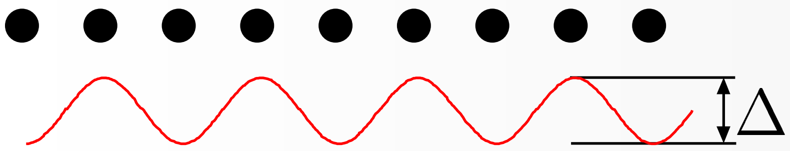
Outline

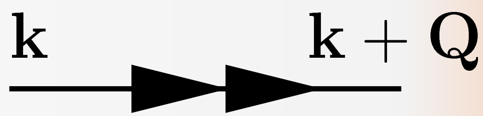
1. Broken symmetry
2. Fermionic functional renormalization group f^2RG
3. Examples

Breaking of a discrete symmetry

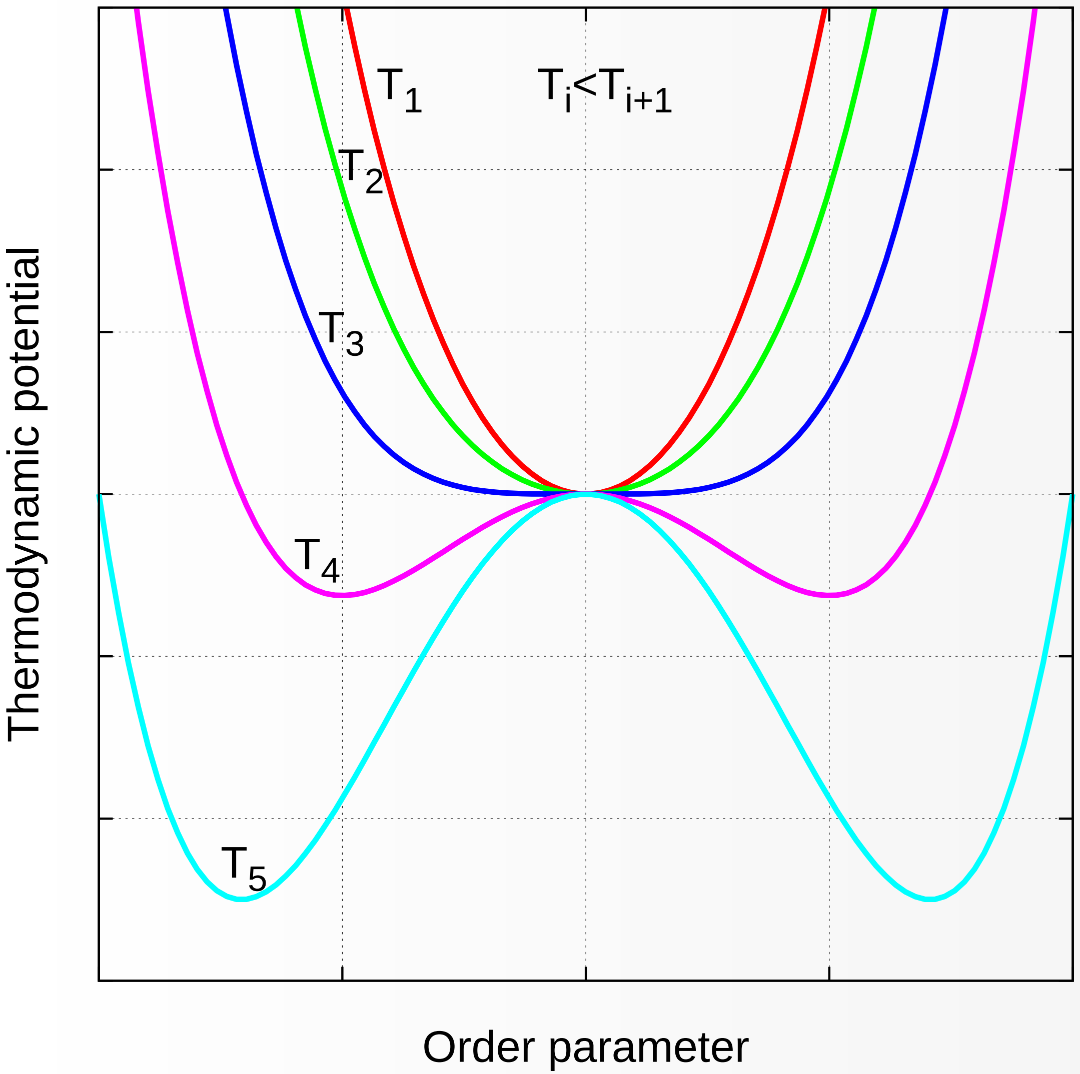
Consider a lattice with electrons in a charge-density wave configuration:



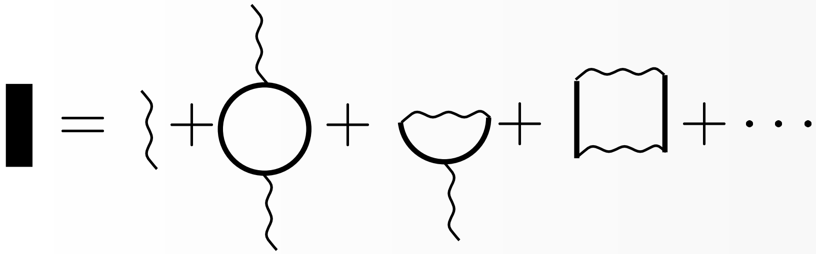
$$H = \sum_{\mathbf{k}} \xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} - \frac{V_0}{N} \sum_{\mathbf{k},\mathbf{k}'} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}+\mathbf{Q}} c^{\dagger}_{\mathbf{k}'} c \; c_{\mathbf{k}'+\mathbf{Q}}$$

- When choosing either the blue or the red configuration, the translational symmetry of the lattice is broken.
- This symmetry is discrete because only a discrete set of choices (blue or red) exists.
- The amplitude Δ of the charge density wave is the order parameter.
- Anomalous Green's function: 

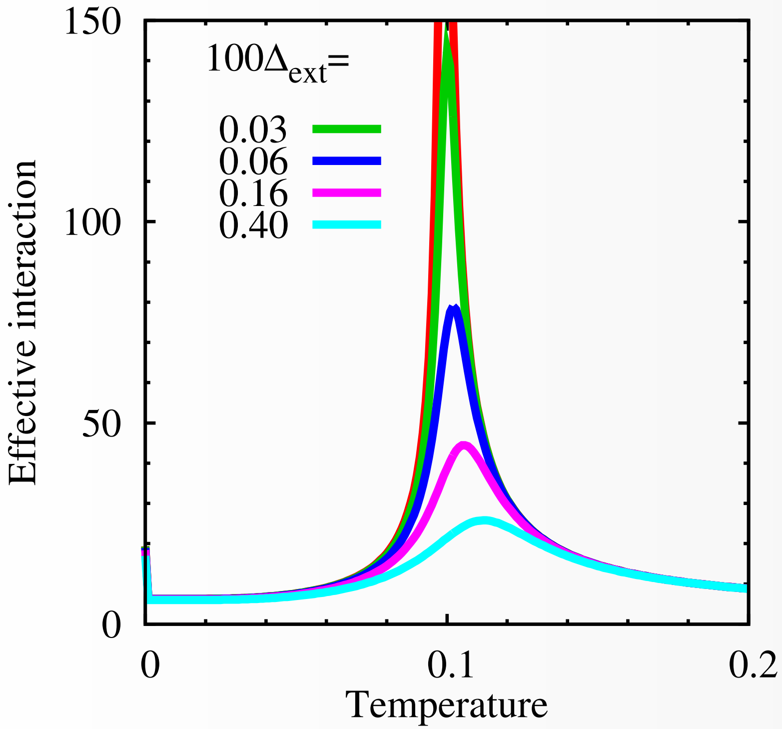
Thermodynamic potential and phases



Effective interaction



Effective interaction: sum of all one-particle irreducible diagrams to which four legs can be attached.



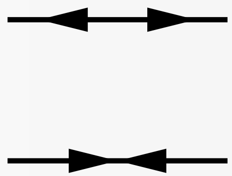
Everything in units of the hopping integral

Breaking of a continuous symmetry

- Example: reduced Bardeen-Cooper-Schrieffer (BCS) model for superconductivity

$$H = H_{\mathrm{kin}} + V_0 \sum_{\mathbf{k},\mathbf{k}'} c^\dagger_{-\mathbf{k}\downarrow} c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}'\uparrow} c_{-\mathbf{k}'\downarrow}$$

- Order parameter: $\Delta_{\mathbf{k}} = \langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \rangle$ is complex.
- Violation of particle-number conservation: anomalous Green´s functions



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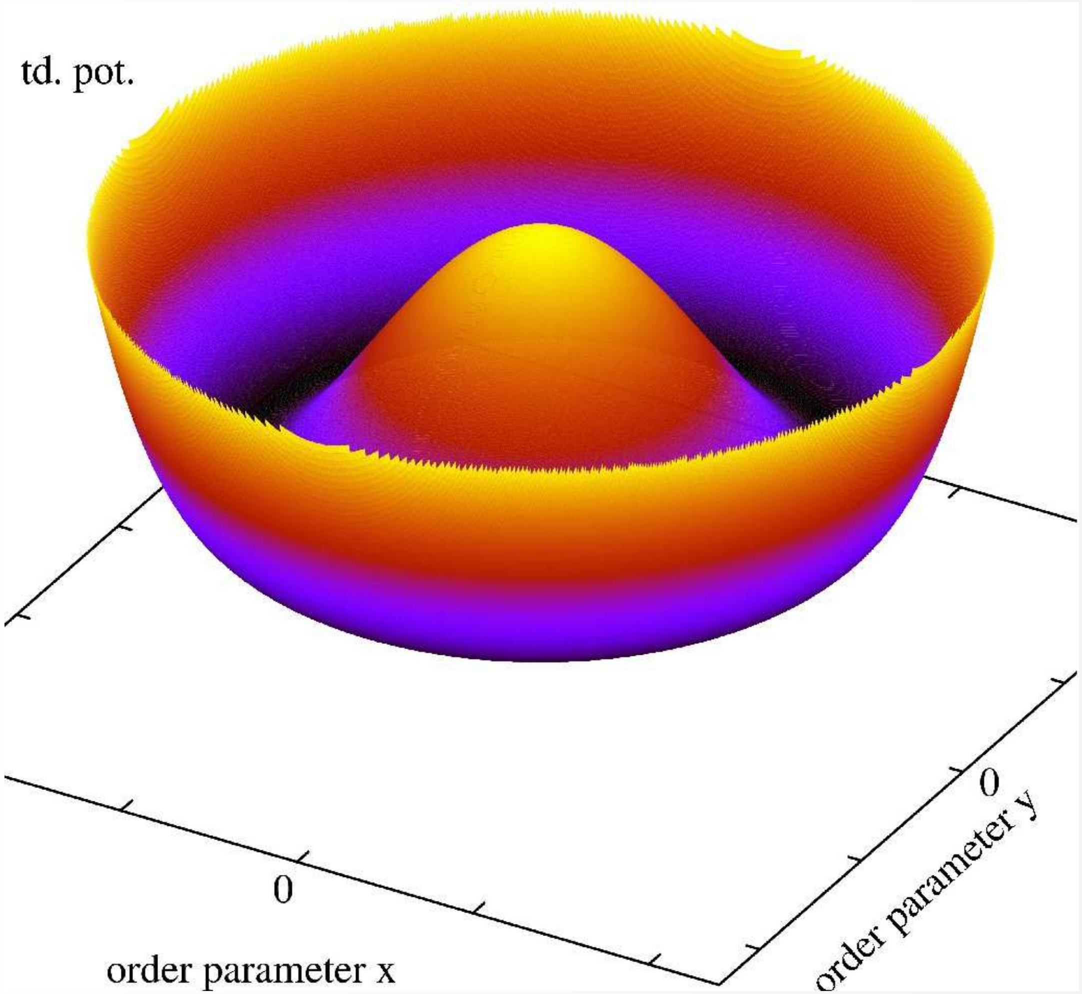
Thermodynamic potential



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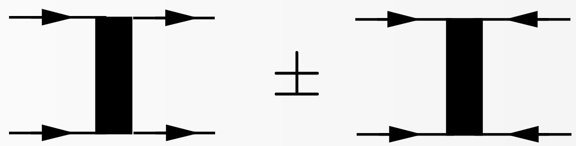
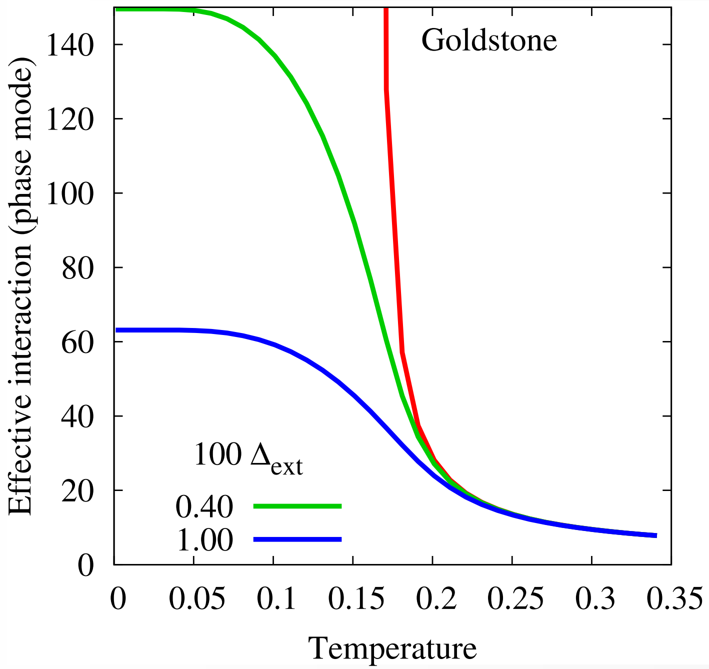
Effective interaction

Two fundamentally different effective interactions due to the anomalous Green's functions:





Everything in units of the hopping integral

Green's function

- The usual Green's function

$$G_\sigma(i\omega_n, \mathbf{k}) = \frac{1}{i\omega_n - \xi_\mathbf{k} - \Sigma_\sigma(i\omega_n, \mathbf{k})}$$

- becomes a matrix

$$\mathbb{G} = \frac{-1}{\omega^2 + (\xi + \Sigma)^2 + \Delta^2} \begin{pmatrix} i\omega + \xi + \Sigma & -\Delta \\ -\Delta^* & i\omega - \xi - \Sigma \end{pmatrix}$$

  by a change of variables which depends on the symmetry that is broken.

- Gapped quasiparticle dispersion $\sqrt{(\xi + \Sigma)^2 + |\Delta|^2}$

The cutoff function

- Dissect the action S (ψ are Grassmann fields, $(\psi, \phi) = \sum_j \psi_j \phi_j$):

$$S(\psi) \stackrel{\text{def}}{=} \nu(\psi, \mathbb{Q}_0 \psi) + V(\psi)$$

- Replace $\mathbb{Q}_0 \to \mathbb{Q}_0 / \chi(\Lambda) =: \mathbb{Q}$.
  1. χ is a matrix of the type of $\mathbb{Q}_0$. The division is element-wise.
  2. Elements of $\mathbb{Q}_0$ for which $\chi(\Lambda) = 0$ do not contribute to the physics.
- Choose χ, Λ_i and Λ_f so that S_{Λ_i} is solvable and $S_{\Lambda_\mathrm{f}} = S$.

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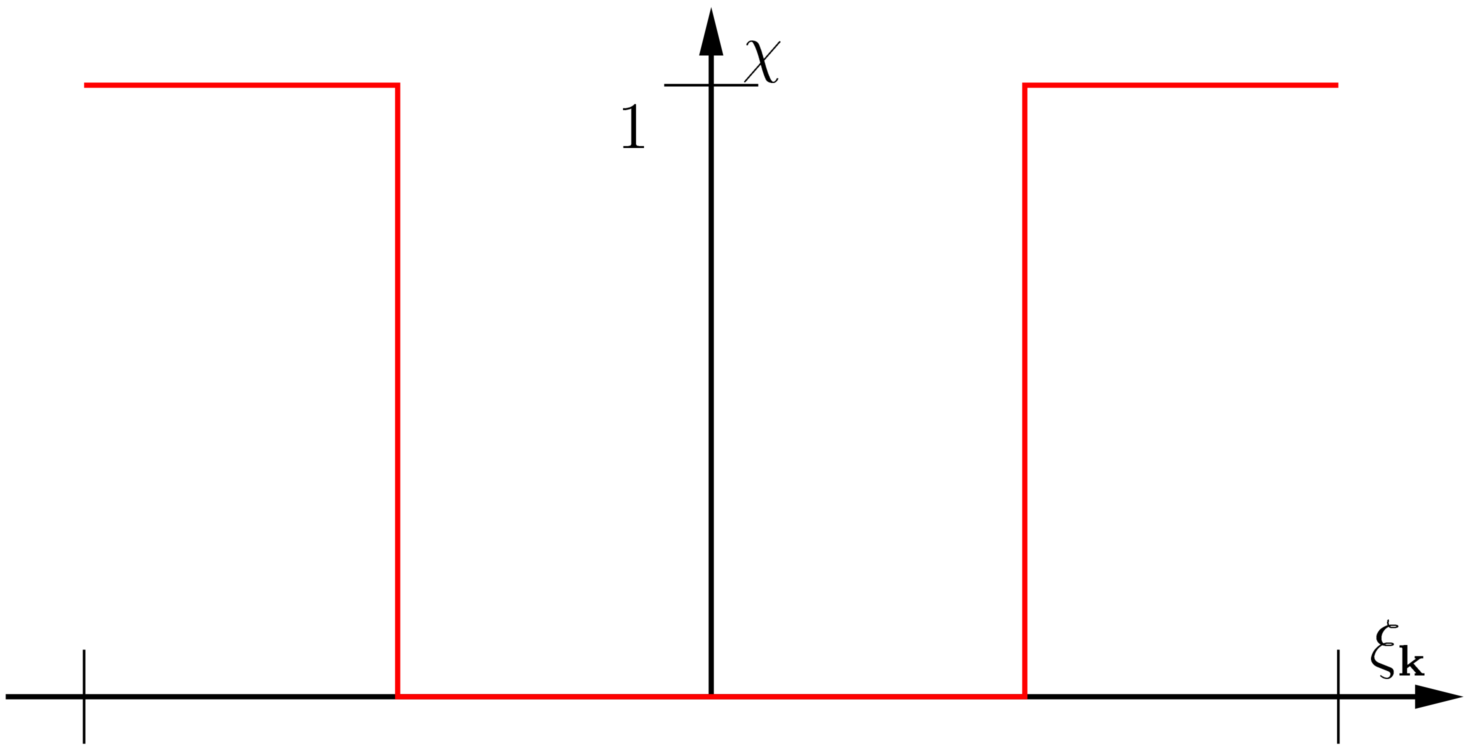
Cutoff function examples

Plot $\mathbf{k}$-dependence of χ.
Sharp momentum cutoff:

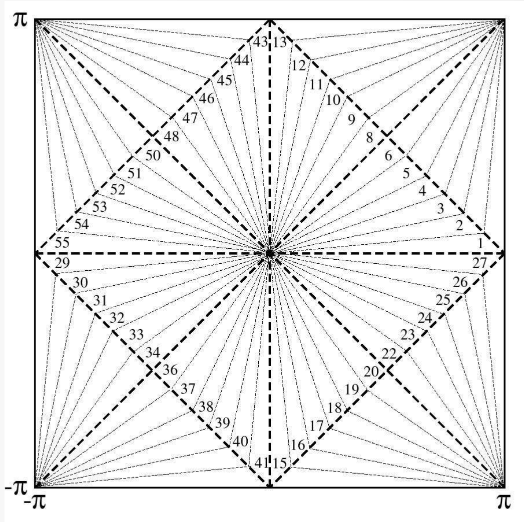


Also popular: interaction, temperature, Matsubara frequency

Some previous work

- Controlling the numerical effort: neglect energy dependence and

  1. project to Fermi surface, patch Brillouin zone
     Zanchi, Schulz, Halboth, Rohe, Metzner, Honerkamp, Fu, Lee . . . , many years



  2. restrict to short-range interactions
     Enss, Andergassen, Metzner, more recently

- Access the symmetry-broken phase: use momentum-cutoff patched f^2RG to obtain effective interactions for a mean-field calculation on a low-energy model. Reiss, Rohe, Metzner cond-mat/0611164

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Derivation of the 1PI RG equation

- Connected Green's functions' generating functional:

$$\exp(-W(H)) = \int \frac{\mathcal{D}\psi}{\det \mathbb{Q}^\nu} e^{-\nu(\psi,\mathbb{Q}\psi)-V(\psi)+(H,\psi)}$$

- Legendre transform $W(H)$:

$$\phi := \frac{\partial W}{\partial H}, \quad H(\phi) := \left(\frac{\partial W}{\partial H}\right)^{-1}(\phi)$$

$$\Gamma(\phi) := W(H(\phi)) - (H(\phi),\phi)$$

Γ is the generating functional of the vertex functions.

- Differentiate (following Salmhofer, Honerkamp 2001):

$$\begin{aligned}\dot{\Gamma}(\phi) &= \partial_\Lambda W(H(\phi)) + (\dot{H}(\phi), \partial_H W(H(\phi))) - (\dot{H}(\phi),\phi) \\ &= \partial_\Lambda W(H(\phi))\end{aligned}$$

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Derivation of the 1PI RG equation

$$\exp(-W(H)) = \det \mathbb{Q}^{-\nu} \int \mathcal{D}\psi e^{-\nu(\psi,\mathbb{Q}\psi)-V(\psi)+(H,\psi)}$$

- LHS: $\partial_\Lambda \exp(-W) = -\dot{W} \exp(-W)$
- Normalization:
  $\partial_\Lambda \det \mathbb{Q}^{-\nu} = \partial_\Lambda \exp \mathrm{Tr}(-\nu \ln \mathbb{Q}) = -\nu \mathrm{Tr}(\dot{\mathbb{Q}}\, \mathbb{Q}^{-1}) \det \mathbb{Q}^{-\nu}$
- Quadratic part of the action:

$$\begin{aligned} & -\nu \det(\mathbb{Q})^{-\nu} \int \mathcal{D}\psi (\psi, \dot{\mathbb{Q}}\psi) e^{\cdots+(H,\psi)} \\ = & -\nu \det(\mathbb{Q})^{-\nu} \int \mathcal{D}\psi (\partial_H, \dot{\mathbb{Q}}\, \partial_H) e^{\cdots+(H,\psi)} \\ = & -\nu (\partial_H, \dot{\mathbb{Q}}\, \partial_H) \exp(-W) \\ = & -\nu \left((\partial_H W, \dot{\mathbb{Q}}\, \partial_H W) + \mathrm{Tr}(\dot{\mathbb{Q}}\, \partial_H^2 W) \right) \exp(-W) \end{aligned}$$

- $\dot{W} = \nu \left(\mathrm{Tr}(\dot{\mathbb{Q}}\, \mathbb{Q}^{-1}) + (\partial_H W, \dot{\mathbb{Q}}\, \partial_H W) + \mathrm{Tr}(\dot{\mathbb{Q}}\, \partial_H^2 W) \right)$

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Derivation of the 1PI RG equation

- $\dot{W} = \nu \left(\mathrm{Tr}(\dot{\mathbb{Q}}\, \mathbb{Q}^{-1}) + (\partial_H W, \dot{\mathbb{Q}}\, \partial_H W) + \mathrm{Tr}(\dot{\mathbb{Q}}\, \partial_H^2 W) \right)$
  Now, eliminate W!
- Remember: $\partial_H W(H(\phi)) = \phi \;\; (\star)$
- Remember: $\Gamma(\phi) = W(H(\phi)) - (H(\phi), \phi)$, therefore

$$\partial_\phi \Gamma(\phi) = \partial_\phi H(\phi) \partial_H W(H(\phi)) - \partial_\phi H(\phi)\phi + H(\phi).$$

  Using $(\star)$: $\partial_\phi \Gamma(\phi) = H(\phi) \;\; (\star\star)$.
- Differentiate $(\star)$ wrt ϕ:
  $\partial_\phi H(\phi) \partial_H^2 W(H(\phi)) = 1\!\!1$
  Plug in $(\star\star)$: $\partial_H^2 W(H(\phi)) = \left(\partial_\phi^2 \Gamma(\phi) \right)^{-1}$
- The 1PI f^{2}RG differential equation:

$$\nu^{-1} \dot{\Gamma} = \mathrm{Tr}(\dot{\mathbb{Q}}\, \mathbb{Q}^{-1}) + (\phi, \dot{\mathbb{Q}} \phi) + \mathrm{Tr} \left[\dot{\mathbb{Q}} \left(\frac{\partial^2 \Gamma}{\partial \phi^2} \right)^{-1} \right]$$

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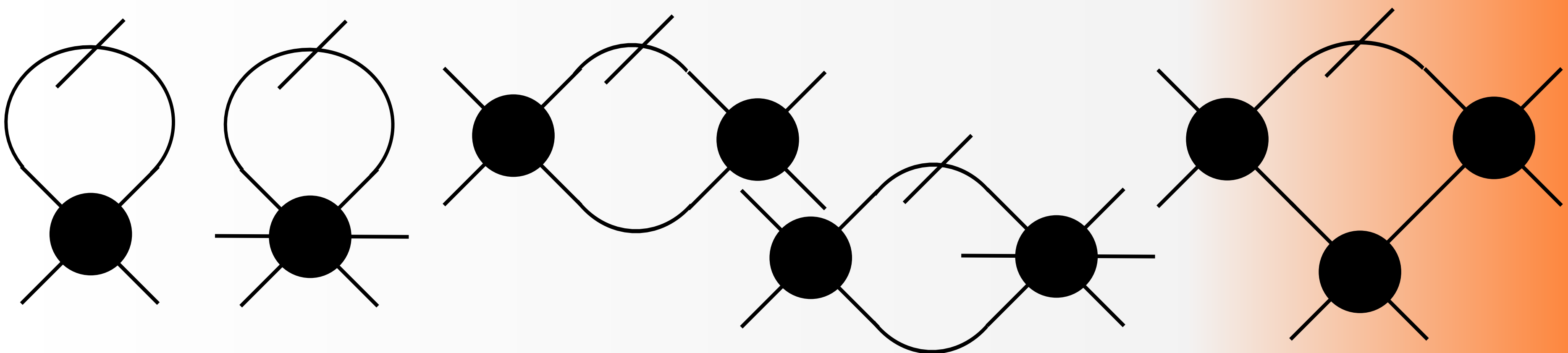
Expansion of Γ and trees

$$\nu^{-1}\dot{\Gamma} = \mathrm{Tr}(\dot{\mathbb{Q}}\,\mathbb{Q}^{-1}) + (\phi, \dot{\mathbb{Q}}\phi) + \mathrm{Tr}\left[\dot{\mathbb{Q}}\left(\frac{\partial^2\Gamma}{\partial\phi^2}\right)^{-1}\right]$$

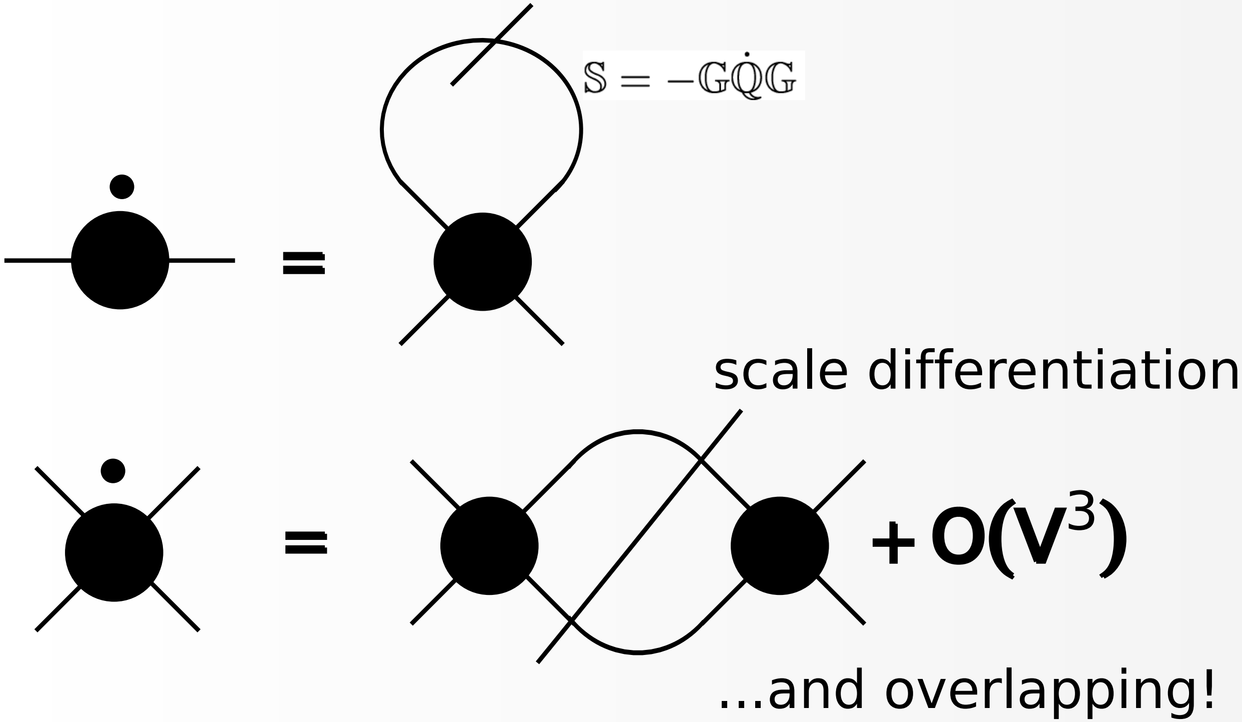
- $\Gamma(\phi) \stackrel{\text{def}}{=} \sum_m \frac{1}{m!}\gamma_m \phi^m$
- Identifying $\gamma_2 = \mathbb{G}^{-1}$, plug the expansion in:

$$\nu^{-1}\dot{\Gamma} = \mathrm{Tr}(\dot{\mathbb{Q}}\,\mathbb{Q}^{-1}) + (\phi, \dot{\mathbb{Q}}\phi) + \mathrm{Tr}\left[\mathbb{G}\dot{\mathbb{Q}}\left(1 + \mathbb{G}\sum_{m\geq 2}\frac{1}{m!}\gamma_{m+2}\phi^m\right)^{-1}\right]$$



Katanin's modification

Katanin 2004

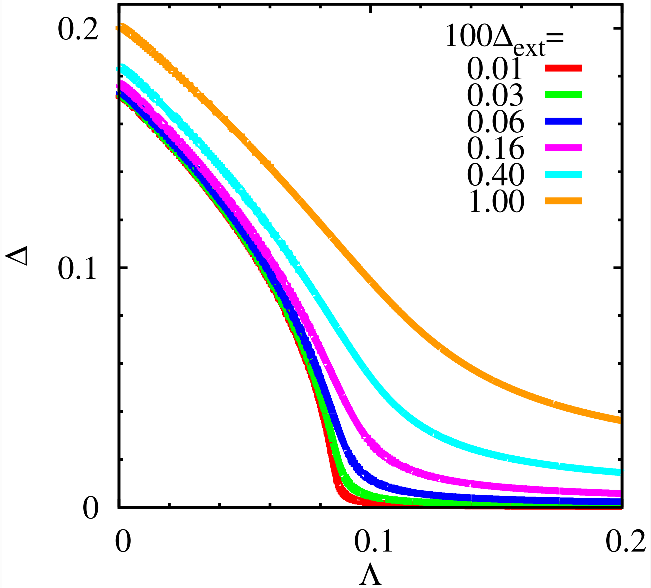


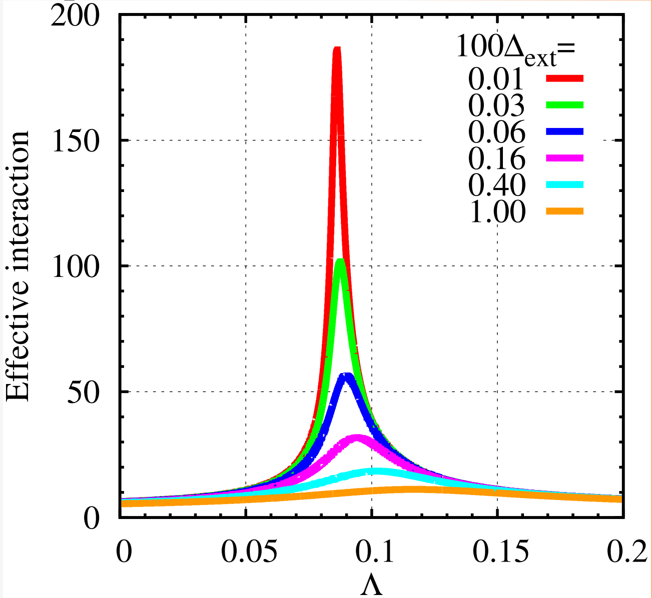
CDW with sharp momentum cutoff

$T = \mu = 0$:

RG, Honerkamp, Rohe, Metzner 2005

Units: hopping integral





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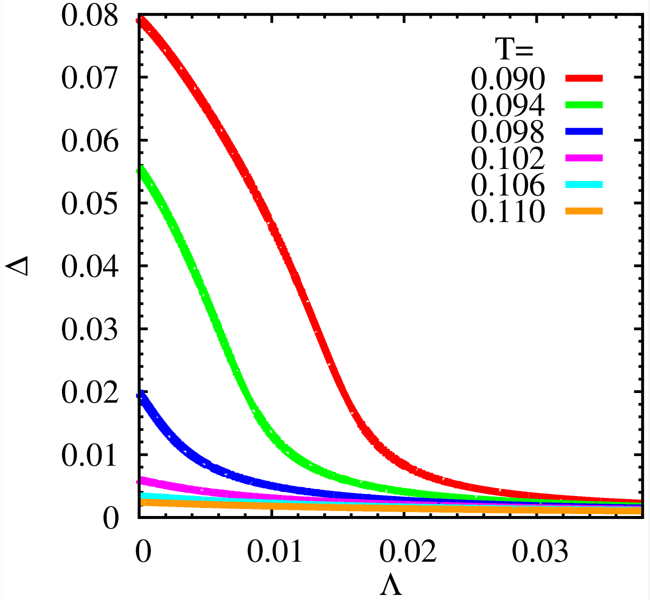
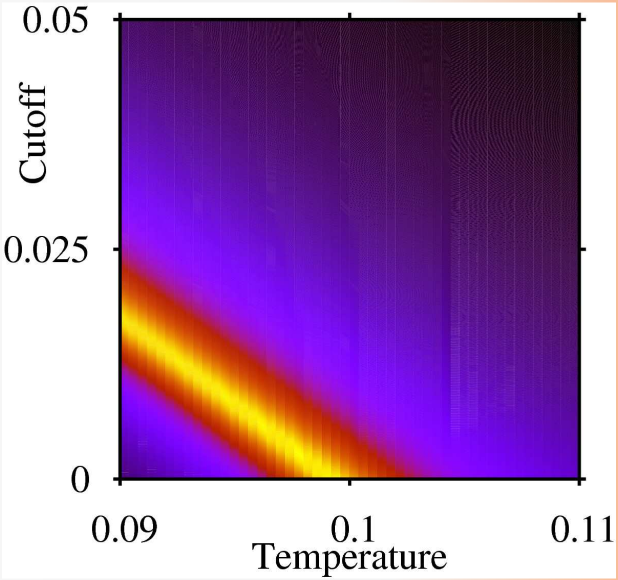
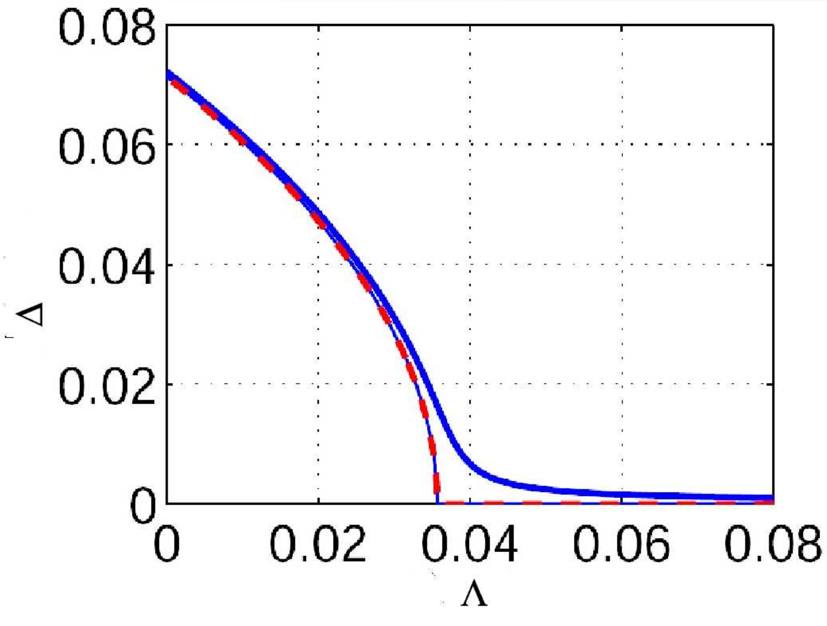
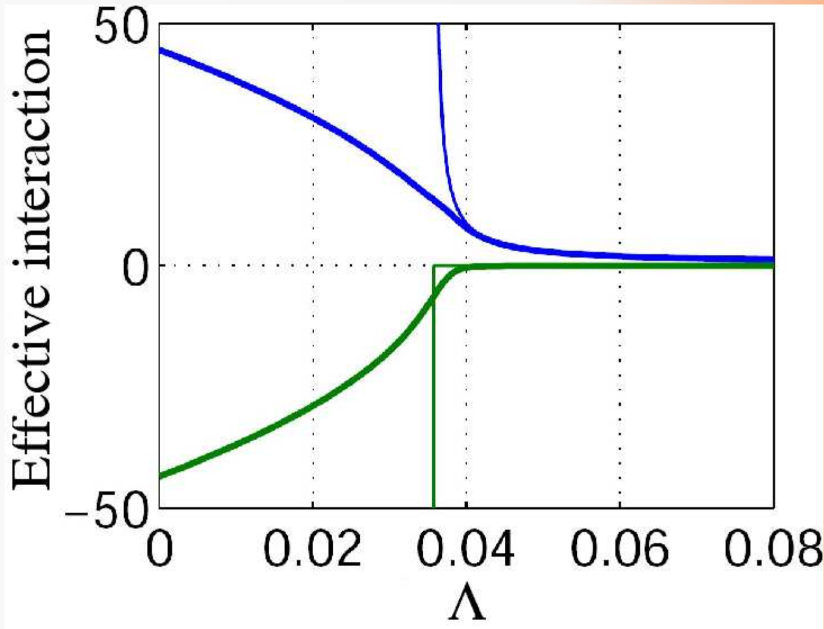




BCS with sharp momentum cutoff, $T = 0$





- Both effective interactions diverge - but not the linear combinations.
- Gap flow is driven by (radial) amplitude mode (addition of the divergent modes).

Everything in units of the bandwidth

Salmhofer, Honerkamp, Metzner, Lauscher 2004

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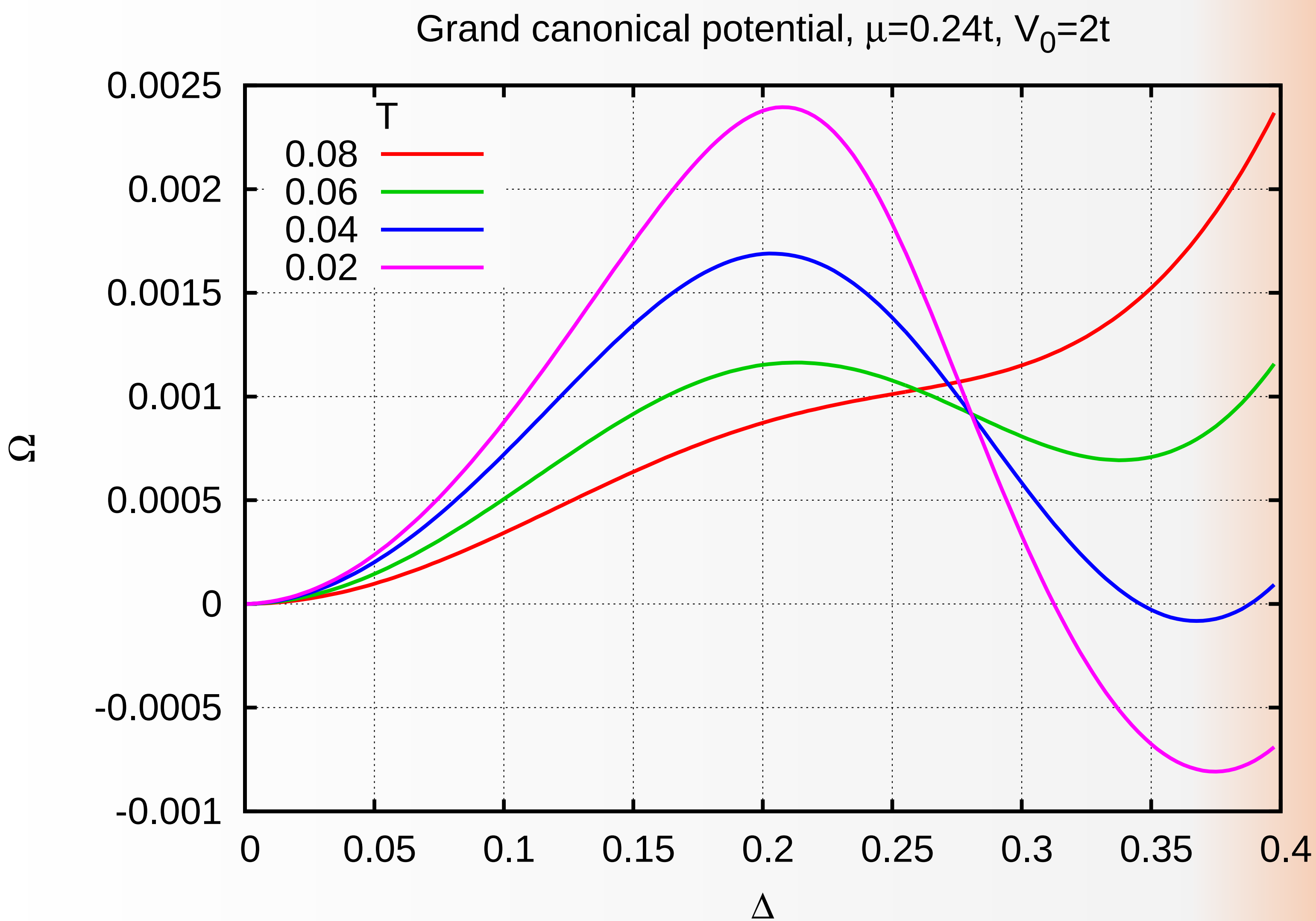
First-order phase transitions



Everything in units of the hopping integral

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Counter terms and interaction flow

- Back to the CDW Hamiltonian.

$$H = \sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - \mu) c^\dagger_{\mathbf{k}} c_{\mathbf{k}} - \frac{V_0}{N} \sum_{\mathbf{k},\mathbf{k}'} c^\dagger_{\mathbf{k}} c_{\mathbf{k}+\mathbf{Q}} c^\dagger_{\mathbf{k}'} c_{\mathbf{k}'+\mathbf{Q}}$$

$$+ \sum_{\mathbf{k}} (\Delta_{\mathrm{c}} - \Delta_{\mathrm{i}}) c^\dagger_{\mathbf{k}+\mathbf{Q}} c_{\mathbf{k}}$$

- To bare propagator
- To initial self-energy

$$\mathbb{G}^{-1} = \frac{1}{\chi} \left(\mathbb{Q} + \Delta_{\mathrm{c}} \sigma_x - \chi \Delta \sigma_x \right) \Rightarrow$$

$$\mathbb{G}_{12} \propto \chi \cdot (\underbrace{\chi \Delta - \Delta_{\mathrm{c}}}_{=: -\Delta_{\mathrm{eff}}})$$

- $\chi = \sqrt{\Lambda}$: Δ_{i} and Δ_{c} cancel *only* at the end of the flow. The initial self-energy can be chosen arbitrarily without changing the physics!

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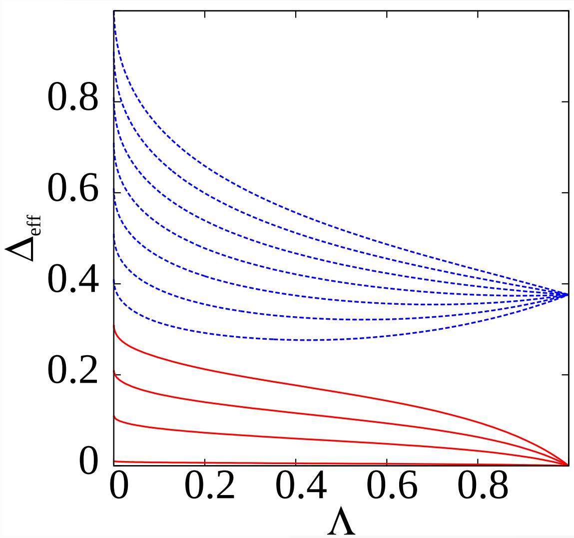
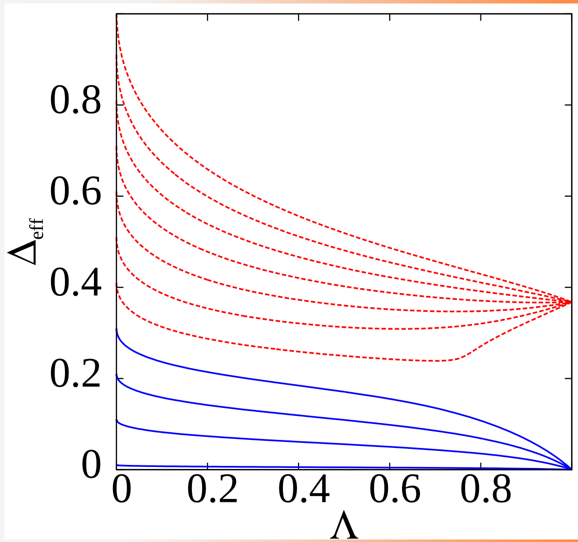
First-order phase transition

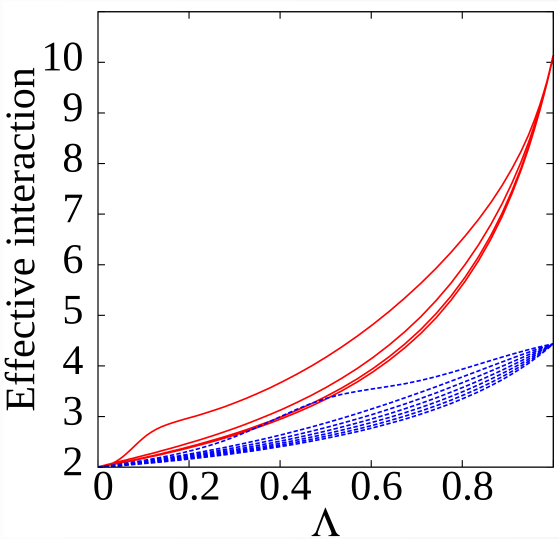
Order parameter
$T < T_\mathrm{t}$
units: hopping integral



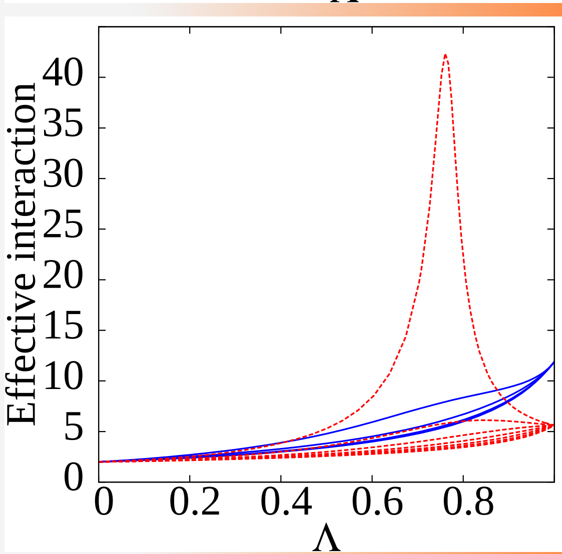
$T > T_\mathrm{t}$



Effective interaction
$T < T_\mathrm{t}$
units: hopping integral



$T > T_\mathrm{t}$



RG, Reiss, Honerkamp 2006

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Conclusions

- The f^2RG is set up as a powerful tool for the study of symmetry breaking.
- Studies of reduced models have been performed:
  1. Second-order phase transition at $T = 0$, broken continuous symmetry (2004)
  2. Second-order phase transition at $T = 0$ and $T > 0$, broken discrete symmetry (2005)
  3. First-order phase transition at $T > 0$, broken discrete symmetry (2006)
- Studies of models with extended momentum structure remain to be done (discretization: patching, expansion, ...?)

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Thank you very much!